Talk: Subdivision and Decomp Numbers.

① KLRW Alg: also known as wgt KLR alg (Webster)
(Maths - Tubbenhauer) or diag Cherednik alg (Bowman, Speyer)

Aim: ① to unify the definition of quiver schur alg
(by Webster, Stroppel)

② Categorify the tensor product of quantum gps

③ In $A_e^{(1)}$, by Webster

W^{ϑ} Either Morita equiv to KLR

or Morita equiv to quiver schur.

depending on weights sum zero or not.

② Def of KLRW Alg.

i) KLRW diagrams:

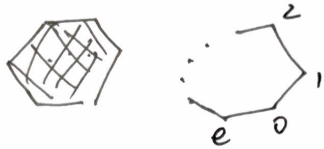
Red string solid - string ghost - string

③ Cellularity

Fix Cartan Data $(A, P, P^\vee, \Pi, \Pi^\vee)$

Consider $A_e^{(1)}$

$\Gamma = (V, E)$



$[0,1] \to \mathbb{R} \times [0,1]$

Smooth map

n, ℓ fixed

fix some data: charge $\rho \in I^\ell$

solid positioning n-tuple $\underline{x} = (x_1, \cdots x_n) \in \mathbb{R}^n$

ghost shift for Γ: $\sigma : E \longrightarrow \mathbb{R}_{\neq 0}$,

wgtning $\qquad \varepsilon \longmapsto \sigma_\varepsilon$

~~charge~~ Red positioning: $\underline{\kappa} = (\kappa_1, \cdots \kappa_\ell) \in \mathbb{R}^\ell$

$\kappa_1 < \cdots < \kappa_\ell$

asympototic if $\kappa_i - \kappa_{i-1} > n \quad \forall\, 2 \leq i \leq \ell$

loading for (Γ, σ)

is a pair $(\underline{\kappa}, \underline{x})$

s.t. $x_i,\ x_j + |\sigma_\varepsilon|,\ \kappa_\varepsilon$ are all different.

Fix n, ℓ, loading $(\underline{\kappa}, \underline{x})$, wgting σ, ρ a charge

a KLRW diagram D of type ~~$(\kappa, x) \to (\nu, y)$~~

$(\kappa, x) - (\nu, y)$

① n solid string located at $s_k(0) = (x_k, 0)$, $s_k(1) = (y_{w(k)}, 1)$ for some permutation $w \in \mathfrak{S}_n$

each carries a residue label i_k

② For $\forall\, \varepsilon: i \rightsquigarrow j$ with $\sigma_\varepsilon > 0$, then draw a ghost of $\forall$ solid i-string, σ_ε units to the right of solid i-string

$\Rightarrow$

$\rightarrow$

with $\sigma_\varepsilon < 0$, then

$\forall$ solid j-string $|\sigma_\varepsilon|$

right

③ Red strings $r_1, \dots r_\ell$, carries label $\rho_1, \dots \rho_\ell$

and connects $(k_1, 0)$. $(k_\ell, 0)$

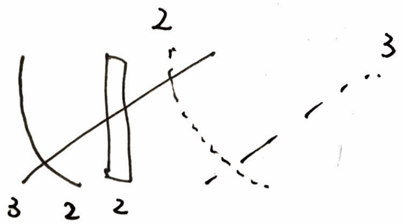
with $(v_1, 1)$ $(v_\ell, 1)$

by straight lines.

④ Solid string can carry dots

· ghost must also have the same dots.

Eg:



Denote the set of diagrams of type $(k, x) \to (v, y)$ with bottom residue by

$W^{(\underline{v}, \underline{y}), \underline{j}}_{(\underline{k}, \underline{x}), \underline{i}}$ ($\underline{j}$ determined by $\underline{i}$)

Rmk: ① Equivalence relation put on KLRW diagram by isotopy: smooth deformation

② $E \circ D$ denote ~~cate~~ usual ~~D~~iagram ~~composition~~ cate.

③

Define $\mathbb{W}^{\rho}_{\beta}(X) = \bigcup_{\underline{x},\underline{y} \in X} \bigcup_{\underline{i},\underline{j} \in I^{\beta}} \mathbb{W}^{\underline{y},\underline{j}}_{\underline{x},\underline{i}}$ $\quad \beta \in Q_n^+$

Take $D \in \mathbb{W}^{\rho}_{\beta}(X)$, $y_r D$ means adding dot to r-th ~~B~~-string

Fix ~~R~~-polynomials

$$Q_{i,j}(u,v) = \begin{cases} u - v & i \to j \\ v - u & j \to i \\ 0 & i = j \end{cases} \quad \text{type } A_e^{(1)}.$$

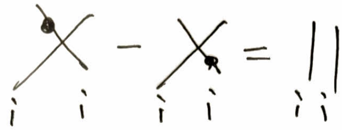
$$Q_{ijk}(u,v,w) = \delta_{ik} \frac{Q_{ij}(u,v) - Q_{kj}(w,v)}{u - w}.$$

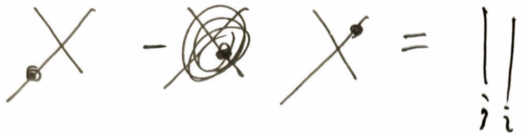
Def: KLRW alg $W^{\rho}_{\beta}(X)$ is (R ~~⊛~~ integral domain). unital associative R-alg by diagrams in $\mathbb{W}^{\rho}_{\beta}(X)$ with multiplication by

$$E \cdot D = \begin{cases} E \circ D & D \in \mathbb{W}^{\underline{y},\underline{j}}_{\underline{x},\underline{i}},\ E \in \mathbb{W}^{\underline{z},\underline{k}}_{\underline{y},\underline{j}} \\ 0 & \text{else} \end{cases}$$

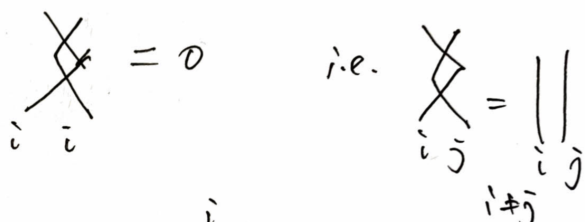
satisfying the following relations:

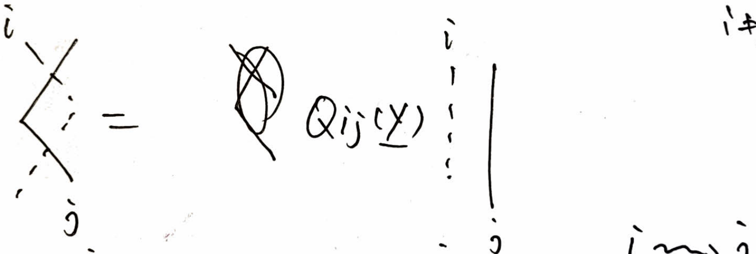
(a) dot sliding relations hold except for the following

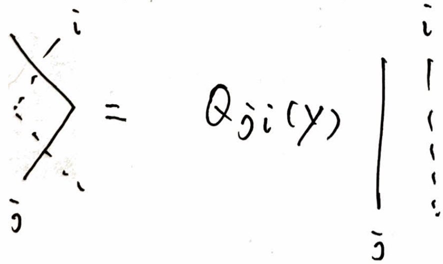


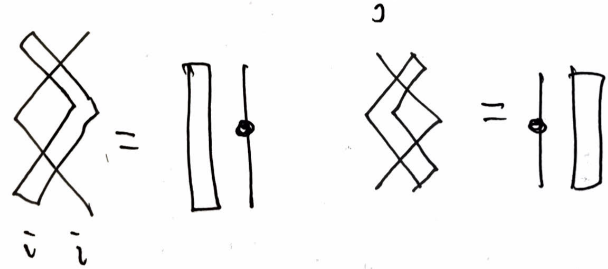


(b) Reidemeister II relations hold except for the following ones:

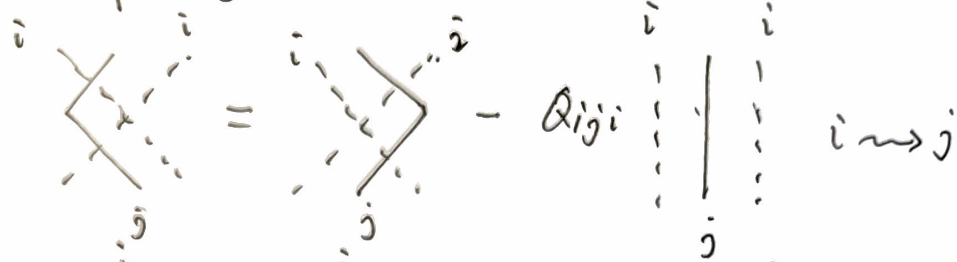


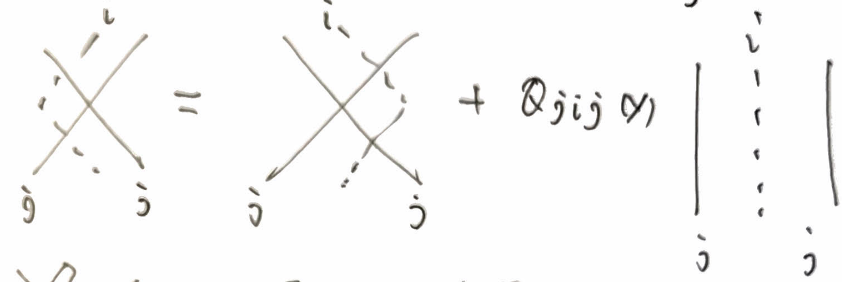


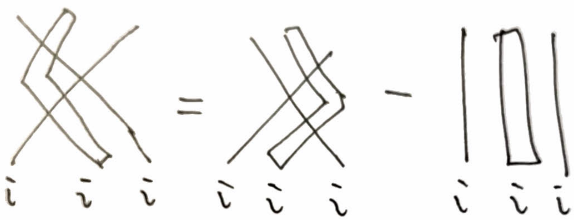




(c) Reidemeister $^{\mathrm{IV}}$ Relations hold except for the following cases:







Set $\mathcal{W}_n^{\rho}(X) = \bigoplus_{\beta \in Q_n^+} \mathcal{W}_\beta^{\rho}(X).$

degree ⊗ : (omit now)

deg | $= 2d_i$. deg ¦ $= 0$.

① homogeneous standard basis.

$$1_{\underline{y},\underline{j}}\, D(w)\, \underbrace{y_1^{a_1}\cdots y_n^{a_n}}_{\underline{y}}\, 1_{\underline{x},\underline{i}} \;\Bigg|\; \begin{array}{l} \underline{x},\underline{y}\in X \\ \underline{i},\underline{j}\in I^{\beta}. \\ w\in \mathfrak{S}_n \\ a_1,\dots,a_n\in \mathbb{Z}_{\geq 0} \end{array}$$

fixed reduced expression for $\forall w$.

② $Z(W^{\rho}_{\beta}(X)) \cong R[Y_1,\dots,Y_n]^{\mathfrak{S}_n}$.

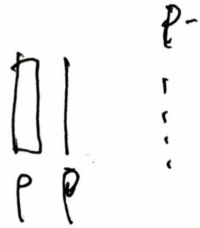
③ ~~graded cellular basis~~.

cyclotomic quotient: $R^{\rho}_{\beta}(X)$

I generated by unsteady idempotent diagrams:

means $\exists$ a solid string can be pulled arbitrary far to the right.

Eg:



unsteady

steady

Prop: $R^{\rho}_{\beta}(X)$ finite dim

④ Cellular basis:

for $\forall$ ℓ-multi $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots \lambda^{(\ell)})$

we form an idempotent 1_λ by

~~$x_{\underline{k}}(\lambda)$~~

$$x_{\underline{k}}(m, r, c) = k_m + (c-r) - \frac{m}{\ell} - (c+r)\varepsilon$$

residue pattern determined by its tabedea
and charge ρ.

dominance order

~~Fix~~
take $X = \bigcup_{\lambda \in \underline{P}_{\ell, n}} x_{\underline{k}}^{A}(\lambda).$

Semi standard tableu λ-tab of type μ

bijection $T: \lambda \to x_{\underline{k}}^{A}(\mu).$

it's semi-std if

① $T(m, 1, 1) \le k_m \quad 1 \le m \le \ell$

② $T(m, r, c) + 1 < T(m, r, c-1)$

③ $T(m, r, c) < T(m, r, c-1) + 1$).

$SStd_{\underline{k}}(\lambda, \mu)$.

Take $S, T \in SStd_{\underline{k}}(\lambda, \mu)$

$\Rightarrow$ $D^a_{ST} := D^*_S y^a 1_\lambda D_T$

where $a = (a_1, \dots a_n) \in \mathbb{Z}^n_{\geq 0}$.

$y^a = y_1^{a_1} \cdots y_n^{a_n}$.

$*$ horizontal flip.

$\Rightarrow$ $B_W = \{ D^a_{ST} \mid \lambda \in P_{\underline{\ell}, n}, S, T \in SStd_{\underline{k}}(\lambda), a \in A^{\underline{k}}(\lambda) \}$

where $A^{\underline{k}}(\lambda) = \{ (a_1, \dots a_n) \in \mathbb{Z}^n_{\geq 0} \mid a_k = 0$ if $\chi^A_{\underline{k}}(\lambda)_k \leq \chi^A_{\underline{k}}(\ell, 1, n) \}$

homogeous

is affine cellular basis

dominance order: $\lambda \trianglelefteq_A \mu$

if $\exists$ bijection $d : [\lambda] \to [\mu]$

s.t. $\chi^A_{\underline{k}}(\alpha) \leq \chi^A_{\underline{k}}(d(\alpha))$ $\forall \alpha \in [\lambda]$

$\Rightarrow$ $B_W = \{ D_{ST} \mid \lambda \in P_{\ell, n}, S, T \in SStd_{\underline{k}}(\lambda) \}$

is homo cellular basis

of cyclotomic KLRW alg $R^P_n(X)$.

Our Aim: ① take ~~cyclotomic~~ KLRW-alg $W^{\rho}_{\beta}(X)$

construct a map (of R-mod)

$$S: W^{\rho}_{\beta}(X) \longrightarrow W^{\underline{\rho}}_{\underline{\beta}}(\bar{X})$$

it induces an isomorphism of graded algebras:

$$S_{\Gamma}: W^{\rho}_{\beta}(X) \xrightarrow{\sim} \cancel{W^{\ell}}\ 1_{\Gamma} W^{\underline{\rho}}_{\underline{\beta}}(\bar{X}) 1_{\Gamma}$$

$1_{\Gamma} J 1_{\Gamma}$

where $J \subset W^{\underline{\rho}}_{\underline{\beta}}(\bar{X})$ is an two-sided ideal

generated by certain relations.

② Take 1_{λ} $\lambda \vdash n$, ℓ-multipartition

~~Consequence~~

$$S(1_{\lambda}) = 1_{\lambda^+} \quad \text{for } \lambda^+ \vdash n+m$$

where $m = \overset{n_i}{\circledcirc}$ or $\overset{n_0}{\circledcirc}$

$$\alpha = \sum n_i \alpha_i$$

$$1_{\lambda^+} \bar{\in} J \quad \forall \lambda^+$$

Consequence: take $\lambda, \mu \vdash n$ ℓ-multi

~~Consider~~ we have $[\Delta(\lambda) : L(\mu)]_q = [\Delta'(\lambda^+) : L'(\mu^+)]_q$ by graded iso.

consider $\Delta^+(\lambda^+)$ and $L^+(\mu^+)$

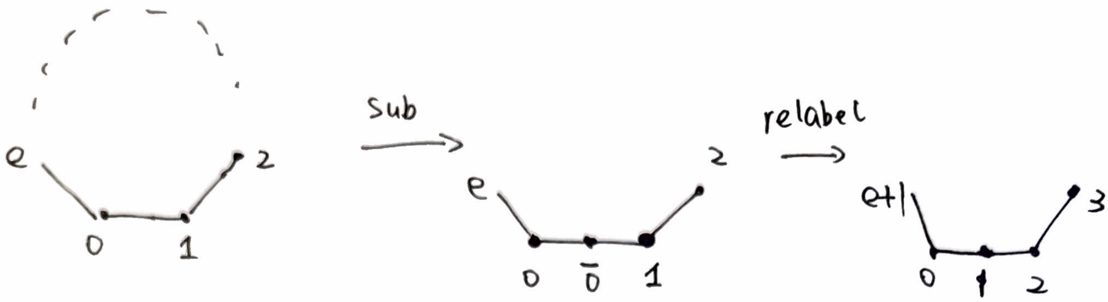
$$\begin{tikzcd} & \lambda^+ \arrow[d, mapsto] \\ \lambda \arrow[ur, mapsto] \arrow[r, mapsto] & \lambda^+ \end{tikzcd}$$

because $L^+(\mu^+)$ never vanishes in J.

$$\Rightarrow [\Delta^+(\lambda^+) : L^+(\mu^+)]_q$$
$$= [\Delta^+(\lambda^+)/\cap : L^+(\mu^+)]_q$$
$$= [\Delta'(\lambda^+) : L'(\mu^+)]_q$$

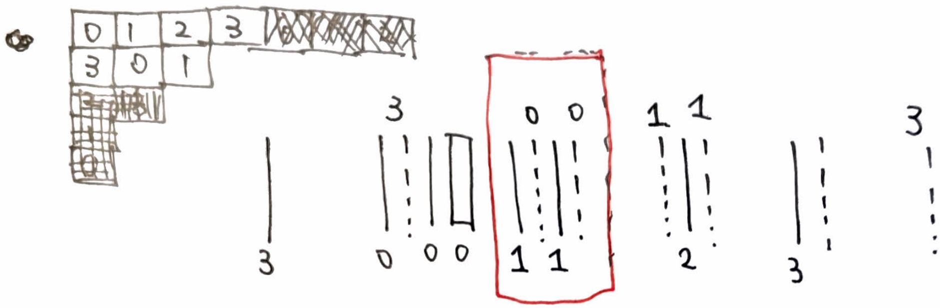
$$\Rightarrow [\Delta(\lambda) : L(\mu)]_q = [\Delta^+(\lambda^+) : L^+(\mu^+)]_q.$$

Remains to define the ~~two~~ map: For simplicity, from now on.
Assume we subdivide at $0 \to 1$ edge



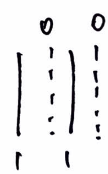
$A_e^{(1)}$ $A_{e+1}^{(1)}$

Define for a basis $D(w) y 1_x$



In the ~~diag~~ idempotents (indexed by tableax)
there are closed $(0-1)$-tuples,

Def: closed $(0,1)$-tuple
with ending residue 1.



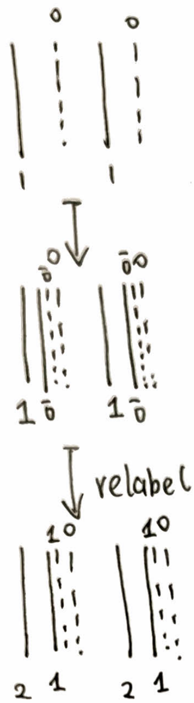
with ending residue 0

closed $(1,0)$-tuple
with ending residue 0

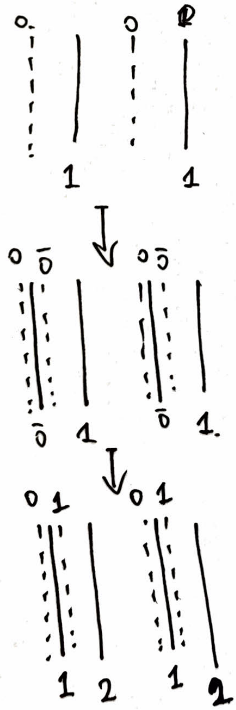
with ending residue 1

maximal if can't be enlarged.

Def: take any maximal closed (0, 1)-tuple



take any maximal closed (1,0)-tuple



① Deal with dots: We don't add dots to the newly added strings

Deal with ~~stand~~ permuted part we extend along the ghost 0.

Problem: this is far from an alg map. there are many relations to be modded out

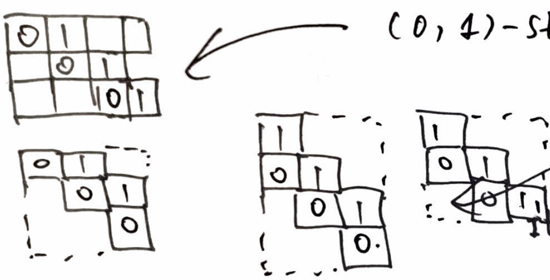
$$\Rightarrow \bar{S} : W^{\rho}_{\beta}(X) \xrightarrow{\sim} 1_{\mathcal{P}} W^{\rho}_{\beta}(\bar{X}) 1_{\mathcal{P}} \Big/ 1_{\mathcal{P}} J 1_{\mathcal{P}}$$

$J = \langle \text{relations} \rangle$ won't discuss here.

Want $S(1_\lambda) = 1_{\lambda^+}$ What is λ^+? ~~we~~

We give two definitions. (Recall we subdivide at $0 \to 1$ only)

For a ~~tableau~~ partition λ together with charge ρ, there are some $(0,1)$-strips and $(1,0)$-strips



It's called maximal if can't be enlarged.

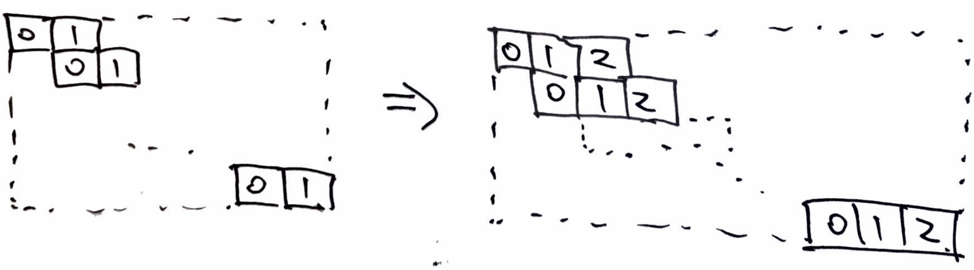
Consider maximal $(0,1)$ & $(1,0)$-strips

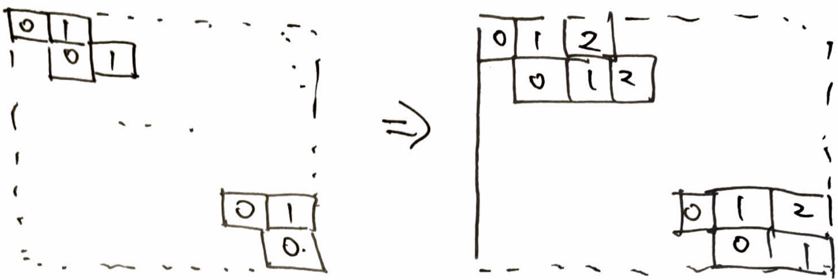
Def: ~~for~~ If $\boxed{0} \Rightarrow \boxed{0|1}$

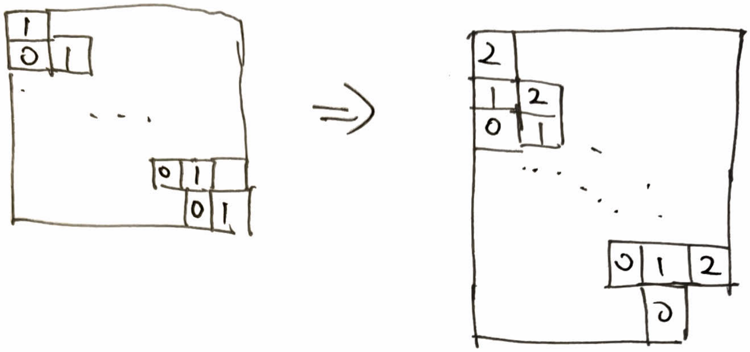
If $\boxed{1} \Rightarrow \boxed{2}$ trivial case.

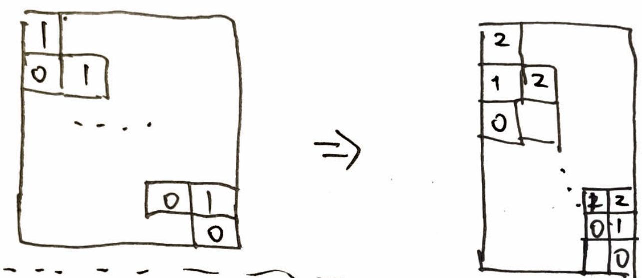
first replace all residue $i \mapsto i+1$ if $i \neq 0,1$.

If



If 

If 

If 

after replacing all those maximal strips we get a new tableau $\rightsquigarrow (\lambda^+, \rho^+)$

Prop 1: λ^+ is a partition

Pf: Every step of replacement still gives a partition. $\square$

Prop 2: ① maximal $(0,1)$-strip ending with $\binom{0}{1}$ corresponds to maximal close $(0,1)$-tuple ending with $\binom{0}{1}$

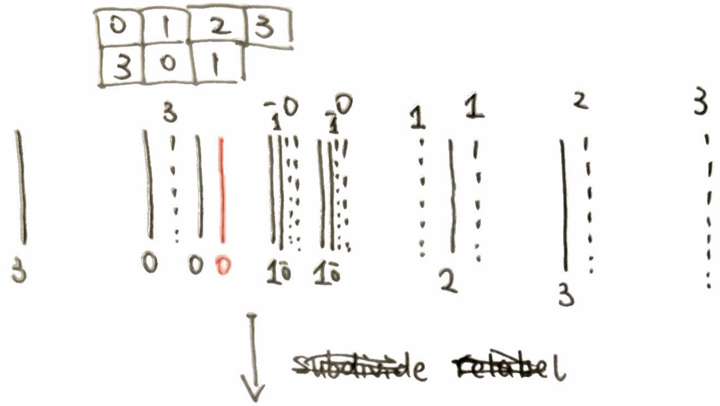
② similarly for maximal $(1,0)$-strips and maximal close $(1,0)$-tuple

Pf: this follows from the combinatorics in type $A_e^{(1)}$

Thm: ~~$1_\lambda^+ = 1_{\lambda^+}$~~ ~~when~~ $S(1_\lambda) = 1_{\lambda^+}$

Pf: there are four cases (locally)

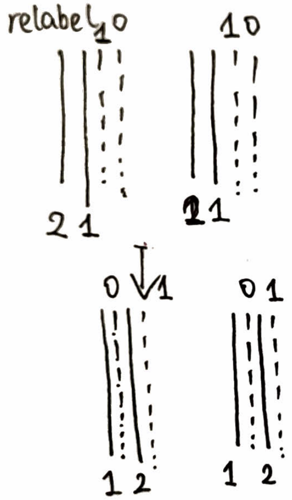
we only show one case by example:



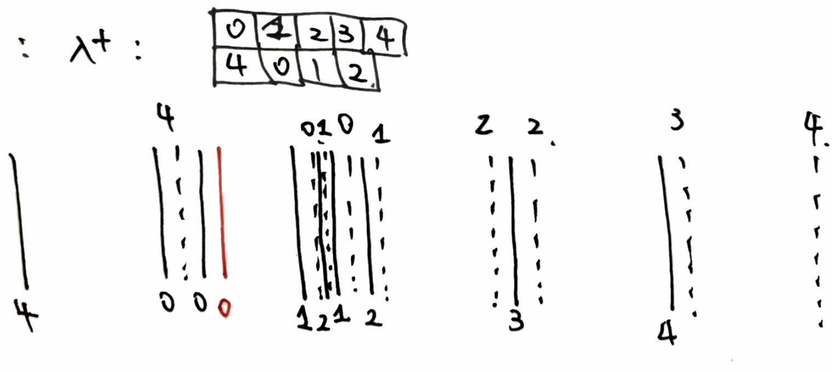
~~subdivide~~ ~~relabel~~

since all other strings remain unchanged (except for labels)

we only draw the close $(0,1)$-tuple here



How about 1_{λ^+}: λ^+:



$\Rightarrow S(1_\lambda) = 1_{\lambda^+}$ $\square$

Alternative Definition of λ^+: Abacus def and Equivalence

Consider 1-partion λ and charge $\rho \in \mathbb{Z}$ [general case is the same]

$$[\lambda]_\rho \longmapsto [\lambda^+]_{\rho^+} \qquad \rho^+ = \begin{cases} 0 & \text{if } \rho = 0 \\ \rho+1 & \text{else.} \end{cases}$$

Let $k(\lambda)$ be the number of max $(1,0)$-strips in $[\lambda]_\rho$

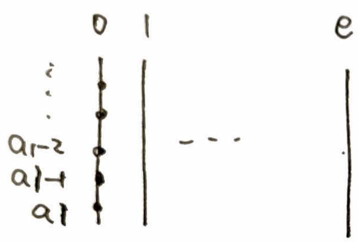
$$\Rightarrow \ell(\lambda^+) = \ell(\lambda) + k(\lambda)$$

For a 1-partition λ, an ~~number~~ integer ρ ~~m, we will take it to~~ be

$$\beta_\rho(\lambda) := (\ \cdots \ \beta_i(\lambda) \ \cdots \ )$$

where $\beta_i(\lambda) = \lambda_i + \rho - i$

For each i $\quad \beta_i(\lambda) = a_i e' + b_i$ where $e' = e+1$



e' runners, call i-runner if it's label is i

$\mathbb{Z}$ levels $\Downarrow$.

put a bead at a_i-level, b_i-runner.

so for (λ, ρ) we have an abacus $Ab(\lambda, \rho)$

we usually truncate upper somewhere

$\forall\, Ab(\lambda, \rho)$ $\exists\, N$ s.t. All positions above level N are occupied by beads

$\exists$ maximal $N_0 \geq N$ all possible N.

$$N_0 = N_0(\lambda, \rho)$$

Conversely, abacus can be defined (independently)

If we put as requirement $\Rightarrow$ $Ab \overset{1:1}{\rightleftarrows} (\lambda, \rho)$

As → we only care about partition, not ~~ρ~~ really ρ

we say two ~~abacus~~ abaci equiv if they give the same partitions.

Two operations won't influence the equivalence

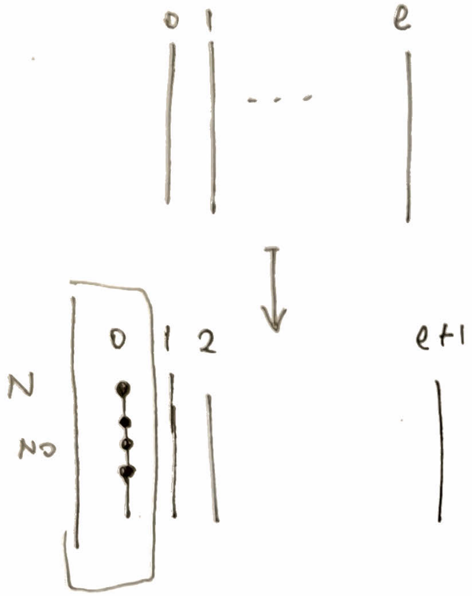
① truncate at different level above N_0 or equal

② moving all beads to the left ~~all~~ or right by some number

How to construct λ^+ by abacus def:

recall for $(\lambda, \rho) \rightsquigarrow Ab(\lambda, \rho) \rightsquigarrow N_0$, $\boxed{k(\lambda)}$

assume we truncate at $N \leq N_0$.



Add $k(\lambda) + 1 + N_0 - N$ beads

or in reality

above N_0 is full

from N_0, add $k(\lambda)+1$ beads on new runner.

$Ab(\lambda, \rho) \mapsto Ab(\lambda_+, \rho_+)$

where $\rho_+ = \rho + N_0 + k(\lambda)$

Later we can show

LM: ① If $\rho = 0$. $N_0 + k(\lambda) = -1$

② If $\rho \neq 0$ $N_0 + k(\lambda) = 0$.

Prop: From $Ab(\lambda,\rho)$ to $Ab(\lambda_+,\rho_+)$

we only add a runner to the left-hand

put beads above or equal to level -1 if $\rho=0$

0 if $\rho\neq 0$

Thm: $\lambda^+ = \lambda_+$

Pf: ① Give explicit formula of λ_+

[Codilize is also possible]

② Check λ^+ ~~satisfies~~ coincides with this formula.

Potential Questions :

① runner removal thm for Ariki-Koike Algs

Compare them

graded subdivision

KLRW $A^{(1)}_e$

$\downarrow$ idemp trunc ?

KLR $A^{(1)}_e$ $\xrightarrow{\text{BK iso.}}$ Ariki-Koike.

? ? $\rightsquigarrow$ full runner removal.

② Algorithm?

③ Other types? $C^{(1)}_e$, $A^{(2)}_{2e}$, $D^{(2)}_{e+1}$